Classical and quantum diffusion in the presence of velocity-dependent coupling

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A generalized system-plus-reservoir model is introduced, which includes four kinds of couplings between the coordinates and velocities of a system and its environment. It is found that the velocity-dependent coupling is not equivalent to a coordinate coupling due to the different power spectra of thermal noise. Harmonic velocity and acceleration noises are proposed which correspond to the coordinate-velocities and velocity-velocities couplings, respectively, if the environmental oscillators are assumed to have a harmonic spectral distribution. Indeed, the velocity-dependent coupling can induce ballistic diffusion of a force-free particle and the mean square velocity depends on the initial preparation. Quantum ballistic diffusion is also presented and its velocity correlation function is found to be unstable at any time. One of real examples of velocity-coordinates coupling is a one-electron atom interacting with the radiation field. A particle moving in a periodic potential shows a nonergodic behavior.

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I. INTRODUCTION

The standard way to describe dissipation in stochastic processes is based on the system-plus-reservoir model, i.e., the Caldeira-Leggett (CL) model [1,2] (also called the independent-oscillator (IO) model [3,4] and the Ullersma's model [5]), where the heat bath consists of quasicontinuous harmonic oscillators and the coupling between the system's coordinate and the reservoir's coordinates is bilinear [6,7]. This model has been applied to a lot of physical situations. One considers usually that the whole system does not have velocity-dependent coupling or force in the theoretical treatment, because some authors [1–3] thought that the velocity coupling was equivalent to the coordinate coupling, namely, the velocity coupling can be transformed into a very similar Lagrangian with a coordinate coupling instead. The most obvious examples of velocity-dependent forces can be found in electromagnetic problems, for example in a superconducting quantum interference device (SQUID) [1,2], where the basic variable, the magnetic flux, is coupled to a quantity with the dimension of electric current. Nevertheless, we have found that their spectra have different forms, which would lead to quite different dynamical behaviors at long times, because the latter is governed by the low frequency part of the power spectrum of a Gaussian noise.

During the last few years, substantial progress has been achieved toward an understanding of anomalous diffusion, which arises from the asymptotical mean square displacement of a force-free particle with the form $\langle x^2(t)\rangle \propto t^{\delta}$, $0<\delta<1$ for subdiffusion, $\delta=1$ for normal diffusion, $1<\delta<2$ for superdiffusion, and $\delta=2$ for ballistic diffusion. It is well known that the non-Ohmic model can describe a rich variety of frequency-dependent damping mechanisms, which arise from a spectral density with the form $J(\omega)=m\gamma_s\omega^s f(\omega/\omega_c)$, where $f(\omega/\omega_c)$ is a cutoff function [6,7]. It is concluded that for $s>2$, the mean square displacement and the variance grow $\propto t^2$. It should be easy to see that we have the following relations: if $s=1$ is for the system's coordinate and environmental coordinates coupling, $s=3$ is for the system's coordinate (velocity) and environmental velocities (coordinates) coupling, and $s=5$ is for the system's velocity and environmental velocities coupling. The latter can occur, for example, when the effect of the blackbody electromagnetic field on a Josephson junction is considered [8,9]. The ballistic diffusion, as the limit of superdiffusion has been theoretically studied in Refs. [6,10–14]; however, dynamical origin and quantum effect remain open.

In this paper, we will discuss a fundamental problem of the theory of Brownian particles: the origin of fluctuations. Our particular focus is on position-velocity and velocity-velocity couplings. These kinds of couplings have not been examined thoroughly so far. Therefore, the present work generalizes previous studies that have been focused on position-position coupling and yields a more comprehensive understanding of the origin of fluctuations. The paper is organized as follows. In Sec. II, we show that one of the dynamical origins of ballistic diffusion might be due to a velocity-dependent bilinear coupling and then discuss classical diffusion induced by the coupling of this kind. It is shown that the results in a long time limit will depend strongly on initial conditions of the particle. In Sec. III, the model is extended to the quantum case. In Sec. IV, the dynamical features of a few systems that exist, velocity-dependent coupling are investigated. A summary is given in Sec. V.

II. THE VELOCITY-DEPENDENT COUPLINGS

The whole Lagrangian under study reads

$$L=\frac{1}{2}m_{\text{eff}}\dot{x}^2-V_{\text{eff}}(x)+\sum_{\alpha=1}^{N}\left(\frac{1}{2}m_\alpha\dot{q}_\alpha^2-\frac{1}{2}m_\alpha\omega_\alpha^2q_\alpha^2\right)+\sum_{\alpha=1}^{N}(C_{00}^{\alpha}xq_\alpha+C_{01}^{\alpha}x\dot{q}_\alpha+C_{10}^{\alpha}\dot{x}q_\alpha+C_{11}^{\alpha}\dot{x}\dot{q}_\alpha), \qquad (1)$$

where x is the coordinate of the system, q_α are the environ-

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mental coordinates described by N independent harmonic oscillators. The coupling between the system and the environmental degrees of freedom is still bilinear. The above model is called the generalized Caldeira-Leggett model (GCL) here, which includes all forms of possible coupling as follows: the coordinate-coordinates (c-c) coupling, the coordinate-velocities (c-v) coupling, the velocity-coordinates (v-c) coupling, and the velocity-velocities (v-v) coupling. In Eq. (1), $m_{\text{eff}}=m-\sum_{\alpha=1}^{\infty}C_{11}^{\alpha 2}/m_\alpha$ and $V_{\text{eff}}(x)=V(x)+\sum_{\alpha=1}^{\infty}C_{00}^{\alpha 2}/(2m_\alpha\omega_\alpha)x^2$, where the additional mass and potential are introduced in order to compensate renormalization of the mass and the potential for v-v coupling and c-c coupling, respectively.

Because the equations of the classical motion for both x and q_α are all linear, $q_\alpha(t)$ can be easily resolved first. Substituting them into the equation of x, a standard generalized Langevin equation (GLE) is obtained

$$m\ddot{x}+m\int_0^t \gamma(t-s)\dot{x}(s)ds+V'(x)=\xi(t), \tag{2}$$

where

$$m\gamma(t-s)=\sum_{\alpha=1}^{N}\left[\frac{(C_{00}^{\alpha}+C_{11}^{\alpha}\omega_\alpha^2)^2}{m_\alpha\omega_\alpha^2}+\frac{(C_{01}^{\alpha}-C_{10}^{\alpha})^2}{m_\alpha}\right]\cos\omega_\alpha(t-s)$$

and

$$\xi(t)=\sum_{\alpha=1}^{N}\Bigg\{[(C_{00}^{\alpha}+C_{11}^{\alpha}\omega_\alpha^2)\tilde{q}_\alpha(0)+(C_{01}^{\alpha}-C_{10}^{\alpha})\tilde{\dot{q}}_\alpha(0)]\cos\omega_\alpha t + \Bigg[-(C_{01}^{\alpha}-C_{10}^{\alpha})\omega_\alpha\tilde{q}_\alpha(0) + \left(\frac{C_{00}^{\alpha}+C_{11}^{\alpha}\omega_\alpha^2}{\omega_\alpha}\right)\tilde{\dot{q}}_\alpha(0)\Bigg]\sin\omega_\alpha t\Bigg\}, \tag{3}$$

where $\tilde{q}_\alpha(0)=q_\alpha(0)-C_{00}^{\alpha}/(m_\alpha\omega_\alpha^2)x(0)$ and $\tilde{\dot{q}}_\alpha(0)=\dot{q}_\alpha(0)+C_{11}^{\alpha}/m_\alpha\dot{x}(0)$. Supposing that the heat bath is initially at the thermodynamic equilibrium in relation to the quantities $\tilde{q}_\alpha(0)$ and $\tilde{\dot{q}}_\alpha(0)$, we have $\langle\tilde{q}_\alpha(0)\rangle=\langle\tilde{\dot{q}}_\alpha(0)\rangle=\langle\tilde{q}_\alpha(0)\tilde{\dot{q}}_\beta(0)\rangle=0$, $\langle\tilde{q}_\alpha(0)\tilde{q}_\beta(0)\rangle=k_BT/(m_\alpha\omega_\alpha^2)\delta_{\alpha\beta}$, and $\langle\tilde{\dot{q}}_\alpha(0)\tilde{\dot{q}}_\beta(0)\rangle=k_BT/m_\alpha\delta_{\alpha\beta}$. Then we get the relation between the correlation function of random force and the damping kernel function, i.e., $\langle\xi(t)\xi(s)\rangle=mk_BT\gamma(|t\text{-}s|)$ [15], where k_B is the Blotzmann constant and T is the temperature of the heat bath.

Equation (3) has been previously derived for $C_{01}=C_{10}=C_{11}=0$ [16], where the friction is generally state dependent and becomes stationary only with an average over the initial probability of starting values. For the exception of a state independent friction in the GLE [17], a discussion of possible pitfalls and open problems was given in Ref. [18]. Nonlinear coupling between the system and the environmental degrees of freedom in a more generic case was considered in Ref. [9].

If all coupling coefficients are assumed to be independent of the frequency of environmental oscillators, the velocity-dependent coupling is not equivalent to a coordinate-coordinates coupling, because they have different spectral densities. As defined in Ref. [1], the relation between the real part of $\tilde{\gamma}(\omega)$ and the spectral density $J(\omega)$ is still retained in the GCL model. The spectral densities of the environmental oscillators are written as $J_{c\text{-}c}(\omega)=\pi/2\sum_\alpha C_{00}^{\alpha 2}/(m_\alpha\omega_\alpha)\delta(\omega-\omega_\alpha)$ for the c-c coupling, $J_{c\text{-}v}(\omega)=\pi/2\sum_\alpha C_{01}^{\alpha 2}\omega_\alpha/m_\alpha\delta(\omega-\omega_\alpha)$ for the c-v(v-c) coupling, and $J_{v\text{-}v}(\omega)=\pi/2\sum_\alpha C_{11}^{\alpha 2}\omega_\alpha^3/m_\alpha\delta(\omega-\omega_\alpha)$ for the v-v coupling. The following relations can be found

$$J_{c\text{-}v}(\omega)=J_{v\text{-}c}(\omega)=\omega^2 J_{c\text{-}c}(\omega),\quad J_{v\text{-}v}(\omega)=\omega^4 J_{c\text{-}c}(\omega), \tag{4}$$

where we have set $C_{00}^{\alpha}=C_{01}^{\alpha}=C_{10}^{\alpha}=C_{11}^{\alpha}$.

Following Eq. (2) and the Kubo second fluctuation-dissipation theorem (FDT) [15], we obtain the exact expression of two-time dynamics for a force-free classical particle. The solution of Eq. (2) can be obtained by means of the Laplace transform technique

$$x(t)=x_0+v_0H(t)+\frac{1}{m}\int_0^t dt' H(t-t')\xi(t'), \tag{5}$$

$$v(t)=v_0\dot{H}(t)+\frac{1}{m}\int_0^t dt'\dot{H}(t-t')\xi(t'), \tag{6}$$

where $H(t)$ is a response function, which is evaluated by the inverse Laplace transform of $\hat{H}(p)$ given by $\hat{H}(p)=[p^2+p\hat{\gamma}(p)]^{-1}$, where $\hat{\gamma}(p)$ is the Laplace transform of the friction memory kernel. The velocity correlation function (VCF) of the particle is given by

$$\begin{aligned}\langle v(t)v(s)\rangle &= \{v_0^2\}\dot{H}(t)\dot{H}(s)+\langle v^2\rangle_{\text{eq}}\int_0^t dt_1\int_0^s dt_2\dot{H}(t-t_1)\\ &\quad\times\dot{H}(s-t_2)\gamma(|t_1-t_2|)\\ &= \{v_0^2\}\dot{H}(t)\dot{H}(s)+\langle v^2\rangle_{\text{eq}}\int_0^t dt_1\int_0^s dt_2[\dot{H}(t-t_1)\\ &\quad\times\dot{H}(s-t_2)*\gamma(s-t_2)+\dot{H}(s-t_1)\dot{H}(t-t_2)\\ &\quad *\gamma(t-t_2)],\end{aligned} \tag{7}$$

where $\{v_0^2\}$ denotes the average with respect to the initial state, $\langle v^2\rangle_{\text{eq}}$ is the mean square velocity of the particle assumed at the equilibrium state, which is determined by the temperature of the heat bath, i.e., $\langle v^2\rangle_{\text{eq}}=k_BT/m$. In addition, "*" denotes the convolution integral.

In the case that $\hat{\gamma}(p)$ is a single valued function of p and the characteristic equation $p+\hat{\gamma}(p)=0$ has only single-double roots, we get

$$\langle v(t)v(s)\rangle=C+S(|t-s|)+A(t,s). \tag{8}$$

The explicit expressions of C, S, and A are given in Appendix A.

The VCF includes of three parts: the constant, stationary, and aging ones. If $\{v_0^2\}=\langle v^2\rangle_{\text{eq}}=k_BT/m$, the aging term vanishes and thus the process is a stationary one at any time, namely, VCF has an invariance behavior of time translation. On the other hand, the velocity variable could become stationary only in a long time limit when $\{v_0^2\}\neq\langle v^2\rangle_{\text{eq}}$. However, the Fourier transform of VCF diverges in both cases, so the Kubo FDT of the first kind [15] is not fulfilled in the ballistic process. If the real parts of all roots are negative, then from the fact that the residue sum of $\hat{\dot{H}}(p)$ in the whole p plane is equal to zero it follows that we have

$$\langle v^2(t\to\infty)\rangle = \langle v^2\rangle_{\text{eq}} + f_c^2(\{v_0^2\} - \langle v^2\rangle_{\text{eq}}), \tag{9}$$

where $f_c=\text{Res}[\dot{H}(0)]$ is given by

$$f_c = \frac{1}{1+\lim_{p\to 0}\left(\dfrac{\hat{\gamma}(p)}{p}\right)}. \tag{10}$$

The condition leading to the stationary velocity correlation depending on the initial velocity preparation of the particle is $\hat{\gamma}(0)=0$ and $\lim_{p\to0}\hat{\gamma}(p)/p$ existing. This also results in the effective friction of the system vanishing, i.e., $\hat{\gamma}(0)=\int_0^\infty \gamma(t)dt=0$. In this case, the asymptotical state of the particle is not a physical equilibrium state and the velocity-dependent coupling may be a dynamical origin of the phenomenon.

Let $\tilde{\gamma}(\omega)$ denote the Fourier transform of the friction memory kernel. Both for c-v (or v-c) coupling and v-v coupling, we have, as $\omega\to0$

$$\begin{aligned}\tilde{\gamma}_{c\text{-}v}(\omega) &\to \frac{-2i\omega}{m\pi}\int_0^\infty d\omega' \frac{J_{c\text{-}c}(\omega')\omega'^2}{\omega'}\frac{1}{\omega'^2}\\ &= \gamma_{c\text{-}c}(0)(-i\omega) = -i\omega c_1,\end{aligned} \tag{11}$$

similarly,

$$\tilde{\gamma}_{v\text{-}c}(\omega) \to -i\omega c_1, \quad \tilde{\gamma}_{v\text{-}v}(\omega) \to -i\omega c_2, \tag{12}$$

where c_1 and c_2 are two finite constants. So the condition of a vanishing effective friction corresponds to the spectral density of thermal noise at zero frequency being equal to zero. We use Eqs. (11) and (12), as $p\to0$

$$\hat{\gamma}_{c\text{-}v}(p), \hat{\gamma}_{v\text{-}c}(p) \to c_1 p, \quad \hat{\gamma}_{v\text{-}v}(p) \to c_2 p. \tag{13}$$

For c-v(v-c) and v-v couplings, $f_c=(1+c_1)^{-1}$, $(1+c_2)^{-1}$, respectively.

It is seen from Eq. (9) that the mean square velocity of the particle in a long time limit depends on its initial velocity distribution. Namely, the mean energy of the force-free particle does not approach a constant and the ergodicity is broken. Thus one needs to introduce an effective temperature as $T_{\text{eff}} := T + f_c^2(T_0 - T)$, where T and T_0 are the temperatures of the heat bath and of the particle at initial time, respectively. Note that the structure of $\langle v(t)v(s)\rangle$ for a general form of $\hat{\gamma}(p)$ is the same as the above case (see Appendix B), for instance, the system will tend with an oscillatory form to its asymptotical state if there exist complex roots in the characteristic equation of the response function.

The mean square displacement of the particle can be obtained exactly when $\hat{\gamma}(p)$ is a single valued function of p and the characteristic equation $p+\hat{\gamma}(p)=0$ only has single-double roots, we have

$$\langle [x(t)-x_0]^2\rangle = a_0 + a_1 t + a_2 t^2 + \Xi(t), \tag{14}$$

where a_0, a_1, and a_2 are three constants, $\Xi(t)$ is a time-dependent decay term. Their explicit expressions are written in Appendix A.

If $f_c\neq0$, which corresponds to the case of pure velocity-dependent bilinear coupling, the highest order for the mean square displacement of the particle is proportional to the square of time at long times, i.e.,

$$\langle [x(t)-x_0]^2\rangle = [f_c\langle v^2\rangle_{\text{eq}} + f_c^2(\{v_0^2\} - \langle v^2\rangle_{\text{eq}})]t^2 = \alpha t^2. \tag{15}$$

The ballistic diffusion occurs. The above expression is always valid for a general form of $\hat{\gamma}(p)$ (see Appendix C). Note that the coefficient α in Eq. (15) cannot be determined by the effective temperature, this means that the present case is ballistic diffusion and not ballistic motion, i.e., $\langle x^2(t)\rangle \neq \langle v^2(t)\rangle t^2$ at long times. Similar results with Eqs. (9) and (15) can be obtained when $\hat{\gamma}(p)$ is a single valued function of p and the characteristic equation $p+\hat{\gamma}(p)=0$ has nonzero multiple roots.

III. QUANTUM BALLISTIC DIFFUSION

In the quantum case, the memory friction function corresponding to the correlation function of thermal noise needs to be replaced by [6,19]

$$\Gamma(t-s) = \frac{\beta\hbar}{m\pi}\int_0^\infty d\omega J(\omega)\coth\left(\frac{\beta\hbar\omega}{2}\right)\cos\omega(t-s), \tag{16}$$

where β is the inverse temperature and $\beta=1/(k_BT)$. The quantum VCF is given by

$$\begin{aligned}\langle v(t)v(s)\rangle = \{v_0^2\}\dot{H}(t)\dot{H}(s) + \langle v^2\rangle_{\text{eq}}\int_0^{\min(t,s)} dt_2\{\dot{H}(s-t_2)\\ \times[\Gamma(t-t_2)*\dot{H}(t-t_2)] + \dot{H}(t-t_2)[\Gamma(s-t_2)\\ *\dot{H}(s-t_2)]\},\end{aligned} \tag{17}$$

and the mean square quantum displacement of the particle is written as

$$\begin{aligned}\langle [x(t)-x_0]^2\rangle &= \{v_0^2\}H(t)^2 + \langle v^2\rangle_{\text{eq}}\int_0^t dt_1\int_0^t dt_2 H(t-t_1)\\ &\quad\times H(t-t_2)\Gamma(|t_1-t_2|)\\ &= \{v_0^2\}H(t)^2 + 2\langle v^2\rangle_{\text{eq}}\int_0^t dt_1 H(t-t_1)\\ &\quad\times[H(t-t_1)*\Gamma(t-t_1)].\end{aligned} \tag{18}$$

The quantum VCF always describes the behavior of a system arriving at the stationary state. A similar calculation with the classical case can be performed. Assuming that both $\hat{\gamma}(p)$ and $\hat{\Gamma}(p)$ are single valued functions of p and the characteristic equation $p+\hat{\gamma}(p)=0$ only has single-double roots, we obtain

$$\langle v(t)v(s)\rangle = C_q + S_q(|t-s|) + A_q(t,s). \tag{19}$$

The explicit expressions of A_q, S_q, and A_q are given in Appendix A.

In comparison with the classical case, $\mathrm{Res}[\hat{\hat{H}}(p_i)] \times \mathrm{Res}[\hat{\hat{H}}(p_j)]$ in terms relating to $\langle v^2\rangle_{\mathrm{eq}}$ are multiplied by $[\hat{\Gamma}(p_i)+\hat{\Gamma}(p_j)]/(-p_i-p_j)$ or $[\hat{\Gamma}(p_i)+\hat{\Gamma}(p_j)]/[-p_i+\hat{\gamma}(p_j)]$, the later form is useful in the derivation of the expression of $\langle v^2(t\to\infty)\rangle$. When the characteristic equation $p+\hat{\gamma}(p)=0$ has two and more nonzero roots (here we view complex conjugate roots as one root), there does not exist a common factor for the aging term, since the correlation function of quantum noise $\Gamma(t)$ does not match to the memory friction function $\gamma(t)$. This means that if we put $\{v_0^2\}$ equal to the mean square velocity at the quantum equilibrium state, the process is still unstable. This is contrary to the classical case. It is not merely appearing in the case of velocity-dependent coupling. This phenomenon can be understood as follows. Corresponding to two and more poles of $\hat{\hat{H}}(p)$, there exist two and more effective modes of the quantum noise. The time distribution of every mode of the noise is collective quantum superposition of the environmental oscillators. The transition between these modes makes the force-free system unstable. In the classical case, the collective superposition reduces to irregular motion of many individual particles, if $\{v_0^2\}=\langle v^2\rangle_{\mathrm{eq}}$, the unstability disappears. If a particle equilibrates with the environment at the beginning, the subsequent motion will violate the Kubo first FDT for the velocity variable in time domain due to the unstability mentioned above.

For the equal-time dynamics at long times, we have

$$\langle v^2(t\to\infty)\rangle = f_q\langle v^2\rangle_{\mathrm{eq}} + f_c^2[\{v_0^2\} - f_q\langle v^2\rangle_{\mathrm{eq}}]. \tag{20}$$

Here $f_c=(1+c)^{-1}$ and the factor f_q is the quantal correction factor given by $f_q=c'/c$, where c' and c are defined by

$$c' = \lim_{p\to 0}\frac{\hat{\Gamma}(p)}{p} = \frac{\beta\hbar}{m\pi}\int_0^\infty d\omega J(\omega)\coth\left(\frac{\beta\hbar\omega}{2}\right)\frac{1}{\omega^2},$$

$$c = \lim_{p\to 0}\frac{\hat{\gamma}(p)}{p} = \frac{2}{m\pi}\int_0^\infty d\omega J(\omega)\frac{1}{\omega^3}. \tag{21}$$

The second moments of velocity and coordinate at the stationary state for the general forms of $\hat{\gamma}(p)$ and $\hat{\Gamma}(p)$ are given in Appendix C. For the system driven by a harmonic velocity noise (HVN) [its spectral density will be determined in Eq. (31)], we have

$$c' = \frac{\beta\hbar\eta\Gamma}{2\omega_1}\left[\coth\left(\frac{\beta\hbar\omega_1}{2}\right) + \coth\left(\frac{\beta\hbar\omega_1}{2}\right)\cot^2\left(\frac{\beta\hbar\Gamma}{4}\right)\right] \times\left[\coth^2\left(\frac{\beta\hbar\omega_1}{2}\right) + \cot^2\left(\frac{\beta\hbar\Gamma}{4}\right)\right]^{-1} - \frac{2i}{m}\sum_{n=1}^{\infty}\frac{J(i\nu_n)}{\nu_n^2}, \tag{22}$$

and

$$c = \eta\Gamma\Omega_0^{-2} \tag{23}$$

with $\omega_1^2=\Omega_0^2-\Gamma^2/4$ and $\nu_n=2n\pi/(\hbar\beta)$. It can be seen that $c'>c$ in general and $c'=c$ when $k_BT\gg\hbar\omega_1$. Note that in the quantum calculation, a cutoff function is necessary for the v-v coupling, but not introduced for the c-v coupling.

If both $\hat{\gamma}(p)$ and $\hat{\Gamma}(p)$ are single valued functions of p and the characteristic equation $p+\hat{\gamma}(p)=0$ has only single-double roots, we yield the mean square displacement of the particle,

$$\langle[x(t)-x_0]^2\rangle = a_{0q} + a_{1q}t + a_{2q}t^2 + \Xi_q(t), \tag{24}$$

where a_{0q}, a_{1q}, a_{2q}, and $\Xi(t)$ are given in Appendix A. As $t\to\infty$, we have

$$\langle[x(t)-x_0]^2\rangle = [f_c^2\{v_0^2\} + f_c(1-f_c)f_q\langle v^2\rangle_{\mathrm{eq}}]t^2. \tag{25}$$

Equations (20) and (25) are also valid for the general forms of $\hat{\gamma}(p)$ and $\hat{\Gamma}(p)$ (see Appendix C).

IV. THERMAL COLORED NOISES CORRESPONDING TO VELOCITY-DEPENDENT COUPLINGS

A. One-electron atom interacting with the radiation field

As an example of the coupling between the system coordinate (velocity) and environmental velocities (coordinates), we consider a one-electron atom interacting with the radiation field [3,8]. The Hamiltonian is written as in the dipole approximation,

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 + U(\mathbf{r}) + \sum_{\mathbf{k},s}\hbar\omega_{\mathbf{k}}\left(a_{\mathbf{k},s}^{+}a_{\mathbf{k},s} + \frac{1}{2}\right), \tag{26}$$

with the vector potential

$$\mathbf{A} = \sum_{\mathbf{k},s}\left(\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V}\right)^{1/2} f_{\mathbf{k}}\hat{e}_{\mathbf{k},s}(a_{\mathbf{k},s} + a_{\mathbf{k},s}^{+}), \tag{27}$$

where $f_{\mathbf{k}}$ is the electron form factor, which is chosen as $f_{\mathbf{k}}^2=\Omega^2/(\omega^2+\Omega^2)$ (here Ω is a large cutoff frequency); $\hat{e}_{\mathbf{k},s}$ is the polarization, and V the volume.

In the limit of large volume for the blackbody cavity, we can use the familiar prescription $\Sigma_{\mathbf{k}}\to V(2\pi)^{-3}\int d\mathbf{k}$ to write the spectral distribution in the form $\tilde{\gamma}(\omega)=(e^2/6\pi m)\int \mathbf{d}k f_{\mathbf{k}}^2\delta(\omega-\omega_{\mathbf{k}})=(2e^2\omega^2/3c^3m)f_{\mathbf{k}}^2$. It has a spectral behavior of the "green" noise [20] (i.e., white noise minus Ornstein-Uhlenbeck noise). For simply, we consider only a one-dimensional version, the memory kernel function is given by

$$\gamma(t) = \gamma_0[\delta(t) - \Omega\exp(-\Omega t)], \tag{28}$$

where $\gamma_0=2e^2\Omega^2/(3c^3m)$ is the static friction strength. In such a case,

$$\hat{\gamma}(p) = \frac{\gamma_0 p}{p+\Omega}. \tag{29}$$

It is easy to see that the motion of the particle is ballistic diffusion since $\hat{\gamma}(0)=0$. This results in $f_c=(1+\gamma_0/\Omega)^{-1}$.

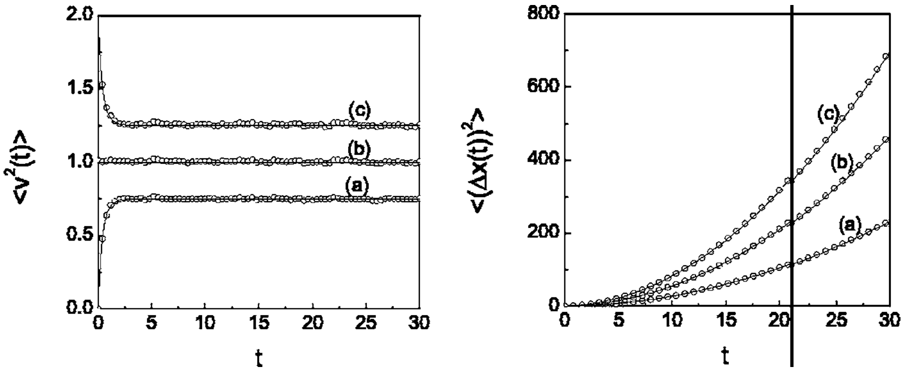

FIG. 1. The second moments of velocity and coordinate of the particle with a memory kernel function (28) at $T=1.0$. The parameters used are $\gamma_0=1.0$, $\Omega=1.0$, (a) $T_0=0$, (b) $T_0=1.0$, and (c) $T_0=2.0$. The open circles are numerical data and the solid lines are theoretical results.

By using the expressions (8) and (14), we obtain

$$\langle v^2(t)\rangle = \{v_0^2\} f_c^2 + \langle v^2\rangle_{\rm eq}(r_f^2+2r_f)f_c^2 + [\{v_0^2\}-\langle v^2\rangle_{\rm eq}] \times[2\Omega^{-2}r_f f_c^2 \exp(-f_c^{-1}\Omega t) + r_f^2 f_c^2 \exp(-2f_c^{-1}\Omega t)] \quad (30)$$

and

$$\begin{aligned}\langle[x(t)-x_0]^2\rangle = &-\langle v^2\rangle_{\rm eq}2\Omega^{-2}r_f f_c^3 + [\{v_0^2\}-\langle v^2\rangle_{\rm eq}]2r_f^2 f_c^4 \\ &+ [\langle v^2\rangle_{\rm eq}2\Omega^{-1}r_f f_c^2 + (\{v_0^2\}-\langle v^2\rangle_{\rm eq})2\Omega^{-1}r_f f_c^3]t \\ &+ [\{v_0^2\}f_c^2 + \langle v^2\rangle_{\rm eq} r_f f_c]t^2 - [\{v_0^2\} \\ &- \langle v^2\rangle_{\rm eq}]2\Omega^{-1}r_f f_c^3 t \exp(-f_c^{-1}\Omega t) \\ &+ \langle v^2\rangle_{\rm eq}2\Omega^{-2}r_f f_c^3 \exp(-f_c^{-1}\Omega t) + [\{v_0^2\} \\ &- \langle v^2\rangle_{\rm eq}]2\Omega^{-2}r_f^2 f_c^4[\exp(-2f_c^{-1}\Omega t) \\ &- 2\exp(-f_c^{-1}\Omega t)], \end{aligned} \quad (31)$$

where $r_f=\gamma_0/\Omega$. By introducing variable transforms [21], the GLE (2) with (28) can be changed into a set of Markovian Langevin equations, which are simulated in order to obtain numerical data. The theoretical expressions (30) and (31) are in good agreement with the simulation results shown in Fig. 1. All quantities plotted here and below are dimensionless, and the initial position of the particle is chosen to be $x_0=0$.

B. The noise spectra of velocity-dependent couplings

Now we consider a case for easy simulation that the environmental oscillators have a harmonic power spectrum and thus the noise corresponding to the coordinate-coordinates coupling is the well-known harmonic noise. Three kinds of spectral densities are given by

$$J_{c\text{-}c}(\omega) = \frac{m\eta\Omega_0^4\omega}{(\omega^2-\Omega_0^2)^2+\Gamma^2\omega^2},$$

$$J_{c\text{-}v}(\omega) = \frac{m\eta\Gamma^2\omega^3}{(\omega^2-\Omega_0^2)^2+\Gamma^2\omega^2},$$

$$J_{v\text{-}v}(\omega) = \frac{m\eta\omega^5}{(\omega^2-\Omega_0^2)^2+\Gamma^2\omega^2}. \quad (32)$$

Here we have set $C_{00}^{\alpha}=C_{01}^{\alpha}=C_{10}^{\alpha}=C_{11}^{\alpha}$. For the c-c, c-v(v-c), and v-v couplings, the values of f_c are equal to 0, $(1+\eta\Gamma/\Omega_0^2)^{-1}$, and $(1+\eta/\Gamma)^{-1}$, respectively. In the underdamped case, we write the memory kernel functions for the c-c, c-v(v-c), and v-v couplings as follows:

$$\gamma_{c\text{-}c}(t) = \frac{\eta\Omega_0^2}{\Gamma}\exp(-\Gamma t/2)\left(\cos\omega_1 t + \frac{\Gamma}{2\omega_1}\sin\omega_1 t\right),$$

$$\gamma_{c\text{-}v}(t) = \eta\Gamma\exp(-\Gamma t/2)\left(\cos\omega_1 t - \frac{\Gamma}{2\omega_1}\sin\omega_1 t\right),$$

$$\gamma_{v\text{-}v}(t) = \eta\delta(t) - \gamma_{c\text{-}c}(t) - \left(1-\frac{2\Omega_0^2}{\Gamma^2}\right)\gamma_{c\text{-}v}(t), \quad (33)$$

where $\omega_1^2=\Omega_0^2-\Gamma^2/4$ and η is the friction of the system corresponding to a thermal white noise. It is easy to prove the facts that $\int_0^\infty\gamma_{c\text{-}c}(t)dt\neq 0$ and $\int_0^\infty\gamma_{c\text{-}v}(t)dt=\int_0^\infty\gamma_{v\text{-}v}(t)dt=0$. Namely, the effective friction of the system vanishes for the velocity-dependent coupling.

A usual second-order colored noise $n_x(t)$, is also called the harmonic noise (HN) [22–24], can be realized from a RLC electric circuit driven by a Gaussian white noise $\zeta(t)$,

$$\ddot{n}_x(t) + \Gamma\dot{n}_x(t) + \Omega_0^2 n_x(t) = \zeta(t). \quad (34)$$

The thermal noise for the c-v or v-c coupling is called the harmonic velocity noise (HVN) [25], i.e., $n_v(t)$. It is the derivative of the noise variable $n_x(t)$ and obeys the Kubo second FDT: $\langle n_v(t)n_v(s)\rangle=mk_BT\gamma_{c\text{-}v}(t\text{-}s)$. The random variable, $n_a(t)$, due to the v-v coupling, which is the second-order derivative of $n_x(t)$ and called the harmonic acceleration noise (HAN) here, which obeys $\langle n_a(t)n_a(s)\rangle=mk_BT\gamma_{v\text{-}v}(t\text{-}s)$. It is noticed that HAN subjected to a Gaussian white noise cannot be procured exactly, because the Gaussian white noise itself cannot be simulated. The integration of the Gaussian white noise can be simulated, therefore, the GLE driven by this noise should be solved numerically.

In Fig. 2, we plot the power spectra $S(\omega)=J(\omega)(m\omega)^{-1}$ of three kinds of colored noises: HN, HVN, and HAN by means of Eq. (32). It is seen that the power spectra of HVN and HAN at zero frequency vanish and that of HN is equal to a constant. It is noticed that the high-frequency part of HVN always decays and it shows a band-passing behavior, however, the power spectrum of HAN approaches a constant when the frequency of noise increases, thus HAN is a kind of

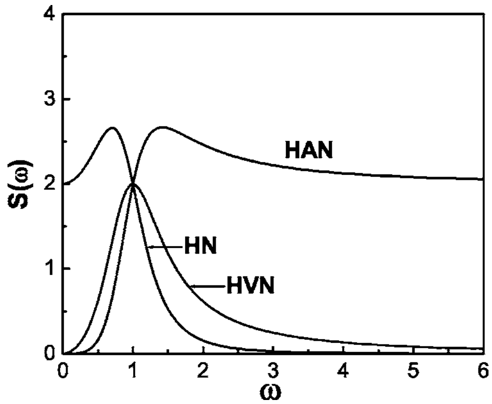

FIG. 2. The power spectra of HN, HVN, and HAN at $\Gamma=1.0$ and $\Omega_0=1.0$.

green noise. These different spectra of thermal colored noises will lead to different dynamical behaviors of a driven-system at long times.

Time-dependent second moments of coordinate and velocity of a force-free particle driven by HVN and HAN are shown in Fig. 3 for various noise parameters, respectively. The numerical data are obtained by simulating a set of Markovian Lengevin equations [21,26] transformed from GLE (2) with the variable $\xi(t)$ corresponding to the noises $n_x(t)$, $n_v(t)$, and $n_a(t)$, which are in agreement with theoretical results calculated by Eqs. (8) and (14). It is known from the expression of factor f_c, $f_c^{(\mathrm{HVN})}>f_c^{(\mathrm{HAN})}$ when $\Omega_0>\Gamma$; however, $f_c^{(\mathrm{HVN})}<f_c^{(\mathrm{HAN})}$ when $\Omega_0<\Gamma$. It is found that a large f_c leads to a large difference of the velocity variance between the stationary state and the equilibrium state if $T_0\neq T$. In the case of $T_0<T$, the larger f_c is, the less both the velocity variance and the prefactor of the mean square displacement at the asymptotical state are. This is due to a fact that the memory effect of the particle to its initial velocity increases with the increase of f_c. For instance, $f_c=1$ is the Newton deterministic mechanics, the behavior of a system at long times is determined completely by its initial condition. On the other hand, when $f_c=0$, the ergodicity is obeyed, the particle forgets its initial state. Thus the stationary state arrives at the equilibrium one in a long time limit.

C. Nonergodicity of a particle moving in a periodic potential

We now consider a Brownian particle subjected to a thermal HVN or HAN in a periodic potential: $U(x)=-U_0\cos x$. The mean energy of the particle is determined by

$$\langle E(t)\rangle = \frac{1}{2}m\langle v^2(t)\rangle + \langle U(x)\rangle, \qquad (35)$$

where $\langle v^2(t)\rangle$ and $\langle U(x)\rangle$ are calculated by simulating a set of non-Markovian Langevin equations transformed from the GLE (2) [21]. The results are plotted in Figs. 4 and 5, where the initial velocity distribution of the particle is chosen to be a Gaussian function with zero mean and variance $\{v_0^2\}=k_BT_0/m$. When the potential barrier height is large enough, the steady mean energy is independent of initial preparation of the velocity and approaches a constant as $t\to\infty$, so the feature of ergodicity emerges. In such a case, the motion of the particle is localized, the bounded potential makes the particle become more disordered, the ergodic state is achieved at long times. When the height of potential barrier is small enough, the motion of the particle is regarded as actively, the value of $\langle E(t)\rangle$ at the stationary state is not equal to a constant and depends on the initial preparation of the particle, thus nonergodicity emerges. This can be understood well from a fact that the memory of the particle to its initial velocity becomes strong when f_c is large in the force-free case. It is seen from Figs. 4 and 5 that initial preparation-dependent evolution becomes observable when $U_0\simeq 0.7$. There is a potential depth threshold $2U_{0m}$ under which the ergodicity is broken. This is similar to the result of Ref. [27].

V. CONCLUSIONS

We have found a condition for which the asymptotical behavior of a force-free particle depends on its initial preparation of velocity, that is the effective friction vanishing or

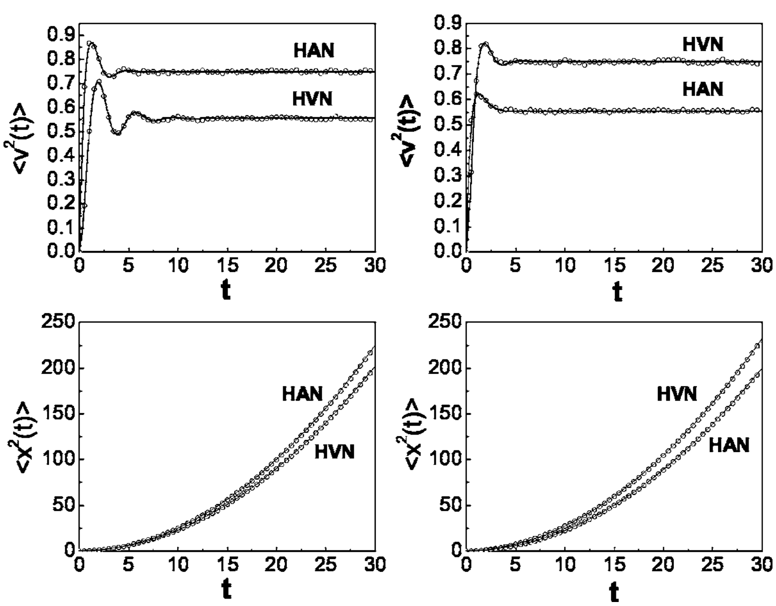

FIG. 3. The second moments of coordinate and velocity of the free particle induced by HAN and HVN for various parameters of the noises at $T=1.0$, $T_0=0.0$, and $\eta=1.0$. The parameters used are $\Gamma=1.0$ and $\Omega_0^2=2.0$ for the left column; $\Gamma=2.0$ and $\Omega_0^2=2.0$ for the right column. The open circles are numerical data and the solid lines are theoretical results.

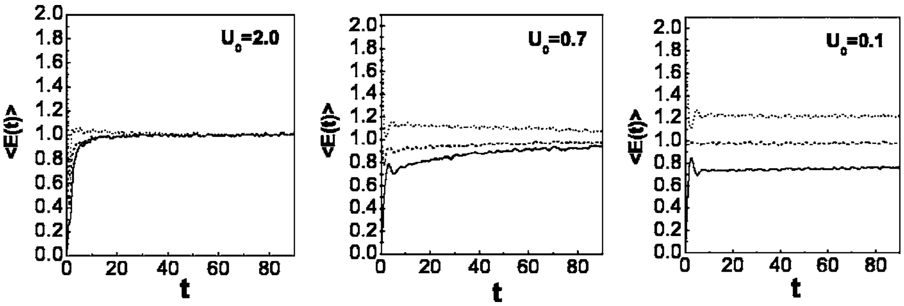

FIG. 4. Time-dependent mean energy of the particle subjected to a thermal HVN. The parameters used are $\Gamma=1.0$, $\Omega_0^2=2.0$, $\eta=1.0$, $T=2.0$, and $T_0=4.0,2.0,0.0$ from top to bottom in each figure.

the spectral density of thermal noise at zero frequency being removed. In this case, the particle exhibits ballistic diffusion in the position space. Indeed, the velocity-dependent coupling might be a dynamical origin of this phenomenon and the velocity-dependent coupling is not equivalent to the position-position coupling because they have different noise spectra. An example for a one-electron atom interacting with the radiation field has been shown to exist such novel phenomenon. Two-time and equal-time classical and quantum dynamics for position and velocity variables have been presented. It has been elucidated that the Kubo fluctuation-dissipation theorem of the first kind is not valid in the quantum case and the system does not exists a unique stationary state. This results in a breakdown of ergodicity.

Two types of Gaussian colored noises which induce ballistic diffusion have been proposed, i.e., the harmonic velocity noise and harmonic acceleration noise, which are the first- and second-order derivatives of the Gaussian harmonic noise, respectively. They have been used as thermal noises to drive a generalized Langevin equation. In addition, the mean energy of a particle moving in a periodic potential has been calculated by using Langevin simulation. The steady mean energy of the particle depends on its initial velocity distribution for a small potential barrier. However, with the increase of potential barrier, the particles are mostly located in the low-energy region, where they experience the quadratic part of the potential. In this case, the mean energy of the particle is equal to a constant and ergodicity can be observed.

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APPENDIX A: SOME QUANTITIES APPEARING IN MAIN TEXT

The quantities appearing in two time velocity correlation function [Eqs. (8) and (19)] are given by

$$C=\{v_0^2\}f_c^2+\langle v^2\rangle_{\rm eq}\sum_{i\neq 0}T_{0i}\delta_i,$$

$$\begin{aligned}S=\langle v^2\rangle_{\rm eq}\sum_{i\neq 0}\delta_i[&\mathrm{Re}\,T_{0i}\cos(\mathrm{Im}\,p_i|t-s|\\&-\mathrm{Im}\,T_{0i}\sin(\mathrm{Im}\,p_i|t-s|)]\exp(\mathrm{Re}\,p_i|t-s|)\\&+\langle v^2\rangle_{\rm eq}\sum_{i,j\neq 0}\delta_{ij}[\mathrm{Re}\,T_{ij}\cos(\mathrm{Im}\,p_i|t-s|)\\&-\mathrm{Im}\,T_{ij}\sin(\mathrm{Im}\,p_i|t-s|)]\exp(\mathrm{Re}\,p_i|t-s|),\end{aligned}$$

$$\begin{aligned}A=\sum_{i\neq 0}[\{v_0^2\}-\langle v^2\rangle_{\rm eq}\delta_i][&\mathrm{Re}\,T_{0i}\cos(\mathrm{Im}\,p_it)\\&-\mathrm{Im}\,T_{0i}\sin(\mathrm{Im}\,p_it)]\exp(\mathrm{Re}\,p_it)+\sum_{i\neq 0}[\{v_0^2\}-\langle v^2\rangle_{\rm eq}\delta_i]\\&\times[\mathrm{Re}\,T_{0i}\cos(\mathrm{Im}\,p_is)-\mathrm{Im}\,T_{0i}\sin(\mathrm{Im}\,p_is)]\exp(\mathrm{Re}\,p_is)\\&+\sum_{i,j\neq 0}[\{v_0^2\}-\langle v^2\rangle_{\rm eq}\delta_{ij}][\mathrm{Re}\,T_{ij}\cos(\mathrm{Im}\,p_it+\mathrm{Im}\,p_js)\\&-\mathrm{Im}\,T_{ij}\sin(\mathrm{Im}\,p_it+\mathrm{Im}\,p_js)]\exp(\mathrm{Re}\,p_it+\mathrm{Re}\,p_js),\end{aligned}\tag{A1}$$

where p_i denote the roots of $p+\hat{\gamma}(p)=0$ with $p_0=0$, $p_i\neq 0$ for $i\neq 0$, and $T_{ij}=\mathrm{Res}[\hat{\dot{H}}(p_i)]\mathrm{Res}[\hat{\dot{H}}(p_j)]$.

The quantities appearing in the classical and quantum mean square displacement of the particle [Eqs. (14) and (24)] are expressed as

$$a_0=\langle v^2\rangle_{\rm eq}\left(-\sum_{i\neq 0}T_{0i}\delta_i\frac{2}{p_i^2}-\sum_{i,j\neq 0}T_{ij}\delta_{ij}\frac{2}{p_i^2}\right)+\sum_{i,j\neq 0}(\{v_0^2\}-\langle v^2\rangle_{\rm eq}\delta_{ij})\frac{T_{ij}}{p_ip_j},$$

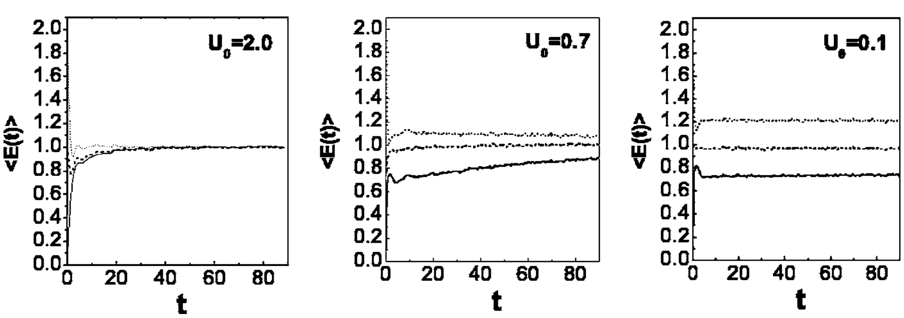

FIG. 5. Time-dependent mean energy of the particle subjected to a thermal HAN. The parameters used are $\Gamma=1.0$, $\Omega_0^2=2.0$, $\eta=1.0$, $T=2.0$, and $T_0=4.0,2.0,0.0$ from top to bottom in each figure.

$$a_1 = \langle v^2\rangle_{\rm eq}\left(-\sum_{i\neq 0} T_{0i}\delta_i\frac{2}{p_i} - \sum_{i,j\neq 0} T_{ij}\delta_{ij}\frac{2}{p_i}\right) - \sum_{i\neq 0}(\{v_0^2\} - \langle v^2\rangle_{\rm eq}\delta_i)T_{0i}\frac{2}{p_i},$$

$$a_2 = \{v_0^2\}f_c^2 + \langle v^2\rangle_{\rm eq}\sum_{i\neq 0} T_{0i}\delta_i,$$

$$\Xi(t) = \sum_{i\neq 0}(\{v_0^2\} - \langle v^2\rangle_{\rm eq}\delta_i)T_{0i}\frac{2}{p_i}t\exp(p_it) + \langle v^2\rangle_{\rm eq}\left(\sum_{i\neq 0} T_{0i}\delta_i\frac{2}{p_i^2} + \sum_{i,j\neq 0} T_{ij}\delta_{ij}\frac{2}{p_i^2}\right)\exp(p_it) + \sum_{i,j\neq 0}(\{v_0^2\} - \langle v^2\rangle_{\rm eq}\delta_{ij})\left(\frac{T_{ij}}{p_ip_j}\right)[\exp(p_it + p_jt) - \exp(p_it) - \exp(p_jt)]. \tag{A2}$$

Note that in the classical case, $\delta_i=1$, $\delta_{ij}=1$, some terms can be incorporated by using $\Sigma_{i\neq 0}{\rm Res}[\hat{\hat{H}}(p_i)]=1-f_c$; in the quantum case, a_{0q}, a_{1q}, a_{2q}, and $\Xi_q(t)$ are the same forms as that in the classical case, but $\delta_i=-\hat{\Gamma}(p_i)/p_i$ and $\delta_{ij}=-[\hat{\Gamma}(p_i)+\hat{\Gamma}(p_j)]/(p_i+p_j)$.

APPENDIX B: A KIND OF EXPRESSION OF $\langle v(t)v(s)\rangle$

The convolution integral $\gamma(t-t_2)*\dot{H}(t-t_2)$ in Eq. (7) is written as

$$\begin{aligned}\gamma(t-t_2)*\dot{H}(t-t_2) &= \frac{1}{2\pi i}\int dp\,\hat{\gamma}(p)\hat{H}(p)\exp[p(t-t_2)]\\ &= \frac{1}{2\pi i}\int dp[1-p\hat{H}(p)]\exp[p(t-t_2)]\\ &= \frac{1}{2\pi i}\int dp[-\hat{\hat{H}}(p)]\exp[p(t-t_2)]\\ &= -\ddot{H}(t-t_2).\end{aligned} \tag{B1}$$

Substituting the above expression into Eq. (7) and performing the partial integration, we have

$$\begin{aligned}\langle v(t)v(s)\rangle &= \{v_0^2\}\dot{H}(t)\dot{H}(s) + \langle v^2\rangle_{\rm eq}\int_t^0 d(t-t_2)[\dot{H}(s-t_2)\\ &\quad\times\ddot{H}(t-t_2) + \dot{H}(t-t_2)\ddot{H}(s-t_2)]\\ &= \{v_0^2\}\dot{H}(t)\dot{H}(s) + \langle v^2\rangle_{\rm eq}\dot{H}(s-t_2)\dot{H}\,(t-t_2)|_{t-t_2=t}^{t-t_2=0},\end{aligned} \tag{B2}$$

for $s>t$. Then

$$\langle v(t)v(s)\rangle = \langle v^2\rangle_{\rm eq}\dot{H}(|t-s|) + [\{v_0^2\} - \langle v^2\rangle_{\rm eq}]\dot{H}(t)\dot{H}(s). \tag{B3}$$

APPENDIX C: CALCULATION OF $\langle v^2(t\to\infty)\rangle$ AND $\langle[x(t\to\infty)-x_0]^2\rangle$

We call $G(t)$ the integration in Eq. (17) (when $s=t$)

$$G(t) = \int_0^t\int_0^t dt_1dt_2\dot{H}(t-t_1)\dot{H}(t-t_2)\Gamma(|t_1-t_2|) \tag{C1}$$

and express $\dot{H}(t-t_2)$ as its inverse Laplace transform,

$$\dot{H}(t-t_2) = \frac{1}{2\pi i}\int dp'\hat{\dot{H}}(p')\exp[p'(t-t_2)]. \tag{C2}$$

The dominant contribution to the integration over t_2 in (C1) comes from a large value of $(t-t_2)$. Because of the exponent, the dominant contribution to the integration over p' comes from a small value of p' in (C2). Therefore, (C2) becomes

$$\dot{H}(t-t_2) = \frac{1}{2\pi i}\int dp'\frac{1}{p'(1+c)}\exp[p'(t-t_2)] = f_c, \tag{C3}$$

then (C1) reads

$$\begin{aligned}G(t) &= f_c\int_0^t dt_1\dot{H}(t-t_1)*\Gamma(t_1-t_2)\\ &= f_c\mathcal{L}^{-1}\left(\frac{1}{p}\hat{\Gamma}(p)\hat{H}(p)\right)\\ &= f_c\frac{1}{2\pi i}\int\frac{1}{p}dp\hat{\Gamma}(p)\hat{H}(p)\exp[p(t-t_1)].\end{aligned} \tag{C4}$$

The dominant contribution comes from $p\to 0$, it is a small region including the origin, can be divided into three parts: the origin part $[p+\hat{\gamma}(p)\to 0]$, the positive imaginary axis part and the negative imaginary axis part. For the origin part, $\hat{\Gamma}(p)=\hat{\Gamma}(-\hat{\gamma}(p))=\hat{\Gamma}(-cp)=\hat{\Gamma}(cp)=c'cp$, so

$$\begin{aligned}\langle v^2(t\to\infty)\rangle &= f_c^2\{v_0^2\} + f_c\left(\frac{c'c}{1+c} + \frac{c'}{1+c} + \frac{c'}{1+c}\right)\langle v^2\rangle_{\rm eq}\\ &= f_c^2\{v_0^2\} + (1-f_c^2)\frac{c'}{c}\langle v^2\rangle_{\rm eq}.\end{aligned} \tag{C5}$$

We express $H(t-t_1)$ in Eq. (18) as its inverse Laplace transform,

$$H(t-t_1) = \frac{1}{2\pi i}\int dp'\hat{H}(p')\exp[p'(t-t_1)], \tag{C6}$$

with a similar analysis to the above, as $t\to\infty$, the dominant contribution to the integration over p' in (C6) comes from a very small value of p', then (C6) becomes

$$H(t-t_1) = \frac{1}{2\pi i}\int dp'\frac{1}{p'^2(1+c)}\exp[p'(t-t_1)] = f_c(t-t_1), \tag{C7}$$

and thus

$$\Gamma(t-t_1)*H(t-t_1)=\frac{1}{2\pi i}\int dp\hat{\Gamma}(p)\hat{H}(p)\exp[p(t-t_1)]=\frac{1}{2\pi i}\int dp\frac{c'p}{p^2(1+c)}\exp[p(t-t_1)]=c'f_c. \tag{C8}$$

The asymptotical form of Eq. (18) reads

$$\langle[x(t)-x_0]^2\rangle=f_c^2\{v_0^2\}t^2+2\langle v^2\rangle_{\mathrm{eq}}f_c^2c'\int_0^t dt_1(t-t_1)=\left[f_c^2\{v_0^2\}+f_c(1-f_c)\frac{c'}{c}\langle v^2\rangle_{\mathrm{eq}}\right]t^2. \tag{C9}$$

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